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# Conservation Laws for Classical and Relativistic Collisions. I

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*Abstract*

On the basis of simple kinematic arguments it is shown that any quantity, depending only on the nature and velocity of a particle, that is conserved in a collision must, in classical mechanics, be of the form $\lambda + \Sigma_i \mu_i v_i + \frac{1}{2}\nu v^2$ or in relativistic mechanics of the form $\lambda + \Sigma_i \mu_i v_i [1 - (v^2/c^2)]^{-1/2} + \nu c[1 - (v^2/c^2)]^{-1/2}$ where $\lambda$, $\mu_i$, and $\nu$ are particle parameters.

## 1. *Introduction*

If two free particles $a$ and $b$ collide, and one or more free particles $c, d, \ldots,$ emerge from the collision, there are quantities depending only on the intrinsic nature of the particles and their velocities that are conserved; that is, it is possible to associate with each particle a quantity $g$ that is a function of its nature and its velocity and that satisfies the condition

$$g[a, \mathbf{v}(a)] + g[b, \mathbf{v}(b)] = g[c, \mathbf{v}(c)] + g[d, \mathbf{v}(d)] + \cdots \tag{1.1}$$

In classical dynamics the mass $m$ and the momentum $m\mathbf{v}$ are always conserved, and the kinetic energy $\frac{1}{2}mv^2$ is sometimes conserved. In relativistic dynamics the momentum $m\mathbf{v}[1 - (v^2/c^2)]^{-1/2}$ and the energy $mc^2[1 - (v^2/c^2)]^{-1/2}$ are always conserved, and the mass $m$ is sometimes conserved.

Depending upon one's point of view, either the conservation theorems are consequences of the fundamental equations of dynamics, or they are the fundamental principles from which all other dynamical results follow. From the first viewpoint the basic principle of dynamics is the fact that particles subjected to the same force will experience the same time rate of change of momentum. From this viewpoint force is a primary quantity. From the second viewpoint the basic principle is the conservation of momentum, and force is simply a secondary quantity, equal by definition to the time rate of change of the momentum. From either viewpoint the critical problem is the definition of momentum.

The fundamental characteristic of the momentum of a particle is that it is a conserved quantity. It is therefore natural to ask, when seeking a definition of momentum, what quantities can possibly be conserved? In this paper and a subsequent paper, I will show on the basis of elementary, plausible, and easily verified kinematic arguments–arguments that do not presuppose any equation of motion but rest on the general concepts of motion–inertial frames, and the law of transformation between inertial frames, that the possible forms of conserved quantities are extremely limited. In particular, in this paper, I will show that if one assumes (1) any quantity that is conserved must be conserved in all inertial frames; and (2) when two identical particles moving along a line with the same speed but in opposite directions collide, then it is possible for the same particles to emerge moving with their original speeds and in opposite directions but along a different line, and that the directions of the line of approach and the line of recession may assume any value; then any quantity, depending only on the nature and velocity of a particle, that is conserved must, in classical mechanics, be of the form

$$g = \lambda + \sum_i \mu_i v_i + \tfrac{1}{2}\nu v^2 \tag{1.2}$$

or in relativistic mechanics be of the form

$$g = \lambda + \sum_i \mu_i v_i [1 - (v^2/c^2)]^{-1/2} + \nu c [1 - (v^2/c^2)]^{-1/2} \tag{1.3}$$

where $\lambda$, $\mu_i$, and $\nu$ are particle parameters, that is, quantities that are constant for a given particle, but whose values vary from particle to particle.

In a later paper I will consider other types of collisions and on the basis of additional assumptions show that the possible forms of conserved quantities can be restricted even further. My reason for breaking the material into two parts rather than combining the result into a single unit was threefold: In the first place, the result of each paper is useful by itself, and I wanted to make either result as accessible as possible. In the second place, the techniques and assumptions in the two papers are somewhat different, and I wished to emphasize this fact. In the third place, I felt that by breaking the material as I have the total result would be easier to understand.

As far as I am aware the general proof given in this paper has never been made, or if it has it is certainly not well known [see for example Smith (1965); Helliwell (1966)]. Tolman (1912) has shown that if in relativistic dynamics there exists a quantity $g$ which is a function of the absolute value $v$ of the velocity $\mathbf{v}$, such that $g(v)$ and $g(v)\mathbf{v}$ are conserved, then $g(v)$ must be proportional to $[1 - (v^2/c^2)]^{-1/2}$. Others [e.g., Lewis and Tolman (1909); Pauli (1921); Weyssenhoff (1935)] have shown that the same result is true if we assume only that $g(v)\mathbf{v}$ is conserved and do not initially assume that $g(v)$ is conserved. Still others [e.g., Ehlers *et al.* (1965)] have shown that the above result is also true if we assume only that $g(v)$ is conserved and do not initially assume that $g(v)\mathbf{v}$ is conserved. Arzelies (1972) has given a brief outline of some of these approaches with appropriate references. The initial restriction to conserved quantities of the form $g(v)$ and $g(v)\mathbf{v}$ is ordinarily justified on the basis that one is seeking to generalize the results of classical mechanics, where it is known that $m$ and $m\mathbf{v}$ are conserved quantities [e.g., Bergmann (1942), Moller (1972)]. If one considers only conserved quantities that are vectors, it can be argued that such a vector must be of the form $g(v)\mathbf{v}$ (Weyssenhoff, 1935; Taylor and Wheeler, 1963). Longmire (1963) has shown, for a classical collision, that if momentum and kinetic energy are conserved, then any other conserved quantity must be a linear combination of the kinetic energy and the components of the momentum. All of the above results are simply special cases of the results obtained in this and the following paper.

## 2. *Conserved Quantities in Classical Collisions*

If two identical particles moving with velocities that in a given inertial frame $S$ are equal in magnitude but opposite in direction collide, then from symmetry we expect as a possible result of this collision the emergence of the same two particles with the same speeds but with the two directions rotated through some common but arbitrary angle. It follows that if there is a conserved quantity $g(\mathbf{v})$ associated with the given type of particle, then we expect it to satisfy the equation

$$g(v\mathbf{n}) + g(-v\mathbf{n}) = g(v\mathbf{n}^*) + g(-v\mathbf{n}^*) \tag{2.1}$$

where $v$ is the speed of each particle, $\mathbf{n}$ is a unit vector parallel to the directions of the precollision velocities, and $\mathbf{n}^*$ is a unit vector parallel to the directions of the postcollision velocities.

If there exists a quantity $g(\mathbf{v})$ that is conserved in one inertial frame, then we expect it to be conserved in all inertial frames. If $S'$ is a frame that is moving with a velocity $-\mathbf{V}$ with respect to $S$, then a particle that is moving with a velocity $\mathbf{v}$ with respect to $S$ will be moving with a velocity

$$\mathbf{v}' = \mathbf{V} + \mathbf{v} \tag{2.2}$$

with respect to $S'$. Converting the precollision and postcollision velocities in equation (2.1) to the values they would have in frame $S'$, and assuming $g$ is still a conserved quantity, we obtain

$$g(\mathbf{V} + v\mathbf{n}) + g(\mathbf{V} - v\mathbf{n}) = g(\mathbf{V} + v\mathbf{n}^*) + g(\mathbf{V} - v\mathbf{n}^*) \tag{2.3}$$

We expect equation (2.3) to be valid for arbitrary values of $v$, $\mathbf{n}$, $\mathbf{n}^*$, and $\mathbf{V}$. This requirement puts severe restrictions on the form of the function $g$. We will now determine what these restrictions are.

If we take the second derivative of equation (2.3) with respect to $v$ and then set $v = 0$, we obtain

$$\sum_i \sum_j n_i n_j g_{ij}(\mathbf{V}) = \sum_i \sum_j n_i^* n_j^* g_{ij}(\mathbf{V}) \tag{2.4}$$

where

$$g_{ij}(\mathbf{V}) \equiv \partial^2 g(\mathbf{V})/\partial V_i \partial V_j \tag{2.5}$$

Since **n** and **n*** in equation (2.4) are arbitrary unit vectors, it follows that the function

$$f(\mathbf{n}) \equiv \sum_i \sum_j n_i n_j g_{ij} \tag{2.6}$$

must be independent of **n**. Only two of the three components $n_1 n_2$, and $n_3$ are independent, since they must satisfy the condition

$$\sum_i n_i^2 = 1 \tag{2.7}$$

If we make use of the method of Lagrange multipliers then we can show that the function $f(\mathbf{n})$ will be independent of our choice of any two of the three components $n_1, n_2, n_3$ if and only if there exists a constant $h$ such that the function

$$F(\mathbf{n}) \equiv \sum_i \sum_j n_i n_j g_{ij} - h\left(\sum_i n_i^2 - 1\right) \tag{2.8}$$

is independent of our choice of any of the quantities $n_1$, $n_2$, and $n_3$. If $F(\mathbf{n})$ is to be independent of our choice of $n_1$, $n_2$, and $n_3$ then the partial derivatives of $F$ with respect to the $n_i$ must vanish. In particular

$$\partial^2 F(\mathbf{n})/\partial n_i \partial n_j = 0 \tag{2.9}$$

Substituting equation (2.8) in equation (2.9) we obtain

$$g_{ij} = h\delta_{ij} \tag{2.10}$$

Recalling that $g_{ij}$ is a function of **V** and noting that $h$, though not a function of **n**, may be a function of **V**, we can rewrite equation (2.10) as follows:

$$g_{ij}(\mathbf{V}) = h(\mathbf{V})\delta_{ij} \tag{2.11}$$

To determine $h(\mathbf{V})$ we note first that

$$g_{ijk} = h_k \delta_{ij} \tag{2.12}$$

where

$$h_k \equiv \partial h(\mathbf{V})/\partial V_k \tag{2.13}$$

The derivative $g_{ijk}$ is equal to the derivative $g_{kji}$, and hence

$$h_i \delta_{kj} = h_k \delta_{ij} \tag{2.14}$$

Choosing $i \neq j = k$ we obtain

$$h_i = 0 \tag{2.15}$$

Since all three partial derivatives of $h$ vanish, it follows that

$$h = \nu \tag{2.16}$$

where $\nu$ is a constant. Substituting equation (2.16) into equation (2.11) we obtain

$$g_{ij}(\mathbf{V}) = \nu\delta_{ij} \tag{2.17}$$

Equation (2.17) provides us with all the second derivatives of $g$. If we solve for $g$ we obtain

$$g(\mathbf{V}) = \lambda + \sum_i \mu_i V_i + \tfrac{1}{2}\nu V^2 \tag{2.18}$$

where $\lambda$, $\mu_1$, $\mu_2$, $\mu_3$, and $\nu$ are constants.

We have shown that if equation (2.3) is true for arbitrary values of $v$, $\mathbf{V}$, $\mathbf{n}$, and $\mathbf{n}^*$ then $g$ must be of the form given by equation (2.18). Conversely it can be shown by direct substitution that if $g$ is of the form given by equation (2.18) then equation (2.3) is satisfied for all values of $v$, $\mathbf{V}$, $\mathbf{n}$, and $\mathbf{n}^*$.

## 3. *Conserved Quantities in Relativistic Collisions*

The same program that we followed for classical collisions can be followed for relativistic collisions. The mathematics can be simplified if instead of dealing with the velocities $\mathbf{v}$ and $\mathbf{V}$ we define the following quantities:

$$\mathbf{u} \equiv \gamma\mathbf{v} \tag{3.1}$$

$$\gamma \equiv [1 - (v^2/c^2)]^{-1/2} \equiv [1 + (u^2/c^2)]^{1/2} \tag{3.2}$$

$$\mathbf{U} \equiv \Gamma\mathbf{V} \tag{3.3}$$

$$\Gamma \equiv [1 - (V^2/c^2)]^{-1/2} \equiv [1 + (U^2/c^2)]^{1/2} \tag{3.4}$$

These are obviously better choices of variables than $\mathbf{v}$ and $\mathbf{V}$, since there is no restriction on the values of $\mathbf{u}$ and $\mathbf{U}$, whereas $\mathbf{v}$ and $\mathbf{V}$ are restricted to values for which $v$ and $V$ are less than $c$.

By the same arguments used in the preceding discussion if $g(\mathbf{u})$ is a conserved quantity then we expect it to satisfy the equation

$$g(u\mathbf{n}) + g(-u\mathbf{n}) = g(u\mathbf{n}^*) + g(-u\mathbf{n}^*) \tag{3.5}$$

If we transform the values of $\mathbf{u}$ to those that they would have in a frame $S'$ moving with a velocity $-\mathbf{V}$ with respect to $S$, noting that the relativistic law of transformation for $\mathbf{u}$ is

$$\mathbf{u}' = \mathbf{u} + \gamma\mathbf{U} + [(\Gamma - 1)(\mathbf{U}\cdot\mathbf{u})\mathbf{U}/U^2] \tag{3.6}$$

we obtain

$$g(\gamma\mathbf{U} + u\mathbf{a}) + g(\gamma\mathbf{U} - u\mathbf{a}) = g(\gamma\mathbf{U} + u\mathbf{a}^*) + g(\gamma\mathbf{U} - u\mathbf{a}^*) \tag{3.7}$$

where

$$\mathbf{a} \equiv \mathbf{n} + [(\Gamma - 1)(\mathbf{U}\cdot\mathbf{n})\mathbf{U}/U^2] \tag{3.8}$$

or equivalently

$$\mathbf{n} \equiv \mathbf{a} - [(\Gamma - 1)(\mathbf{U}\cdot\mathbf{a})\mathbf{U}/(\Gamma U^2)] \tag{3.9}$$

Since $\mathbf{n}$ is a unit vector, the components of $\mathbf{a}$ are not all independent but must satisfy the auxiliary condition

$$(c^2 + U^2)\left(1 - \sum_i a_i^2\right) + \sum_i \sum_j U_i U_j a_i a_j = 0 \tag{3.10}$$

which is obtained by squaring equation (3.9) and making use of the definition of $\Gamma$ given by equation (3.4).

It follows from the preceding development that if $g(\mathbf{u})$ is a conserved quantity then it must satisfy equation (3.7) for arbitrary choices of $u$ and $\mathbf{U}$ and also for arbitrary choices of $\mathbf{a}$ and $\mathbf{a}^*$ consistent with the constraint condition equation (3.10). This requirement puts severe restrictions on the form of $g$. We shall now determine what these restrictions are.

If we take the second derivative of equation (3.7) with respect to $u$, remembering that $\gamma$ is a function of $u$, and then set $u = 0$ we obtain

$$\sum_i \sum_j a_i a_j g_{ij}(\mathbf{U}) = \sum_i \sum_j a_i^* a_j^* g_{ij}(\mathbf{U}) \tag{3.11}$$

where

$$g_{ij}(\mathbf{U}) \equiv \partial^2 g(\mathbf{U})/\partial U_i \partial U_j \tag{3.12}$$

By the same argument we used in the preceding section equation (3.11) will be true if and only if there exists a constant $h$ such that the function

$$F(\mathbf{a}) \equiv \sum_i \sum_j a_i a_j g_{ij} - h\left\{(c^2 + U^2)\left(1 - \sum_i a_i^2\right) + \sum_i \sum_j U_i U_j a_i a_j\right\} \tag{3.13}$$

is independent of our choice of the variables $a_1$, $a_2$, and $a_3$. It follows that

$$\partial^2 F(\mathbf{a})/\partial a_i \partial a_j = 0 \tag{3.14}$$

Substituting equation (3.13) into equation (3.14) we obtain

$$g_{ij}(\mathbf{U}) = h(\mathbf{U})[(c^2 + U^2)\delta_{ij} - U_i U_j] \tag{3.15}$$

To determine $h(\mathbf{U})$ we note that

$$g_{ijk} = h_k[(c^2 + U^2)\delta_{ij} - U_i U_j] + h[2U_k\delta_{ij} - U_j\delta_{ik} - U_i\delta_{jk}] \tag{3.16}$$

The derivative $g_{ijk}$ is equal to the derivative $g_{kji}$ and hence

$$h_k[(c^2 + U^2)\delta_{ij} - U_i U_j] + h[2U_k\delta_{ij} - U_j\delta_{ik} - U_i\delta_{jk}] = h_j[(c^2 + U^2)\delta_{ki} - U_k U_i] + h[2U_j\delta_{ki} - U_i\delta_{kj} - U_k\delta_{ij}] \tag{3.17}$$

Choosing $i \neq j = k$ we obtain

$$(h_i/h)[(c^2 + U^2) - U_j^2] + (h_j/h)U_i U_j = -3U_i \tag{3.18}$$

Interchanging $i$ and $j$ in equation (3.18) we obtain

$$(h_i/h)U_iU_j + (h_j/h)[(c^2 + U^2) - U_i^2] = -3U_j \tag{3.19}$$

Solving equations (3.18) and (3.19) for $(h_i/h)$ we obtain

$$h_i/h = -3U_i/(c^2 + U^2) \tag{3.20}$$

Equation (3.20) provides us with the three partial derivatives of ln $h(\mathbf{U})$. If we solve for $h$ we obtain

$$h = \nu(c^2 + U^2)^{-3/2} \tag{3.21}$$

where $\nu$ is a constant. Substituting equation (3.21) into equation (3.15) we obtain

$$g_{ij} = \nu(c^2 + U^2)^{-3/2}\,[(c^2 + U^2)\delta_{ij} - U_iU_j] \tag{3.22}$$

Equation (3.22) provides us with all the second derivatives of $g$. If we solve for $g$ we obtain

$$g(\mathbf{U}) = \lambda + \sum_i \mu_i U_i + \nu(c^2 + U^2)^{1/2} \tag{3.23}$$

where $\lambda$, $\mu_1$, $\mu_2$, and $\mu_3$ are constants. If we replace **U** by its definition in terms of **V** we obtain

$$g(\mathbf{V}) = \lambda + \sum_i \mu_i V_i[1 - (V^2/c^2)]^{-1/2} + \nu c[1 - (V^2/c^2)]^{-1/2} \tag{3.24}$$

We have shown that if equation (3.7) is true for arbitrary values of $u$ and **U**, and also arbitrary values of **a** and **a*** consistent with the constraint condition (3.10) then $g$ must be of the form given by equation (3.24). Conversely it can be shown by direct substitution that if $g$ is of the form given by equation (3.24) then equation (3.7) is satisfied for all values of $u$ and **U** and also all values of **a** and **a*** consistent with the constraint condition (3.10).

## 4. *Conclusion*

In this paper it has been shown on the basis of primarily kinematical arguments that the possible forms of velocity-dependent conserved quantities are extremely limited. In a subsequent paper the problem will be considered further.

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